

# Quantitative Reasoning

Introductory article

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*Quantitative reasoning refers to human and non-human animals' sensitivity to and ability to represent different types of numerical information, including both discrete and continuous quantities, and to perform mental operations over these representations.*

## INTRODUCTION

The ability to mentally represent number and quantity is one of the human species' greatest assets. It has led to the development of our formal system of mathematics, which has enabled the advancement of our species throughout much of its recent history. As an abstract, logical system, mathematics allows us to represent both physical and abstract entities. This article will discuss how numerical information is represented in the human mind, and how the mind uses these representations to carry out basic numerical computations such as simple addition and subtraction. Recent research suggests that our sensitivity to number may be an evolved capacity and one that we share with other animal species.

## THE NEUROPSYCHOLOGY OF NUMBER PROCESSING

The psychologist Stanislas Dehaene formulated a model of number processing, which currently dominates the study of numerical cognition. According to this model, humans' number knowledge is comprised of three representational systems, occupying distinct areas of the brain. Although these systems represent numerical information differently, they are interconnected, and function together, under normal circumstances, to support the numerical abilities of humans.

The 'verbal number' system is located in the language area of the left hemisphere of the brain.

It represents numerical information linguistically, including both spoken and written number words. It also represents well-learned arithmetical facts, such as addition and multiplication tables learned by rote, and supports multidigit calculations, which require both the recall of memorized facts and the visuospatial representation of numbers. The 'visual Arabic numeral' system, located in the occipito-temporal region of both hemispheres, represents the visual forms of Arabic numerals. Note that not all cultures use the Arabic form of numbers in their written language. Presumably this area of the brain would represent the written form of numerals whether it be Arabic or otherwise. However, as the known data implicating this area of the brain have been tested only with Arabic numerals, we will continue to refer to it here as the 'visual Arabic numeral' system. A 'magnitude' system, located in the inferior parietal cortex regions of both hemispheres, represents numerical information in the form of magnitudes. It gives us our sense of quantity – the meanings of numbers – and therefore supports our ability to compare two numbers, perform approximate calculations, and so on. Empirical findings support the idea that such processes rely on a magnitude representation of number. For example, in the 'distance effect', adults become faster at saying which of two numbers is larger as the proportionate difference between the numbers increases. This is consistent with a magnitude representation of number, which predicts (as explained below) that as the proportionate difference between two numbers becomes smaller, it becomes more difficult to compare them.

Since these three systems are interconnected, activating the representation of, say, 'seven' in the verbal number store automatically activates both the visual Arabic form '7' and the magnitude representation of the number 7. Nevertheless,

neuropsychological evidence clearly indicates that these systems are both functionally and anatomically distinct: they are disrupted by damage to different areas of the brain, and if one system is damaged, the functioning of the remaining two systems is often preserved.

Studies of patients with brain damage support the existence of distinct neural circuitry underlying each of these different systems. For example, there are patients with left-hemisphere injuries affecting their language areas who show deficiencies in understanding number words and performing simple rote calculations such as 'eight plus nine' or 'six times four'. However, they remain able to recognize Arabic numerals, make magnitude judgments (e.g. whether 3 is larger than 7), and even compute approximate results of numerical calculations – for example, judging correctly that ' $2 + 2 = 9$ ' is false, while erroneously judging that ' $2 + 2 = 5$ ' is correct. In contrast, there are patients with damage to the inferior parietal cortex (where the magnitude number system is located) who can recognize written and spoken number words and Arabic digits, but have lost their sense of magnitude. These patients can recite arithmetic facts, such as multiplication tables, but cannot say, for example, whether 3 is larger or smaller than 7.

The fact that patients with left-hemisphere damage may show impaired ability to recognize and produce verbal number words, yet still be able to judge approximate magnitudes and visually recognize Arabic numerals, suggests that the damaged (but not the preserved) abilities rely on areas of the brain within the left hemisphere. Similarly, the fact that patients with damage to the inferior parietal cortex show impaired performance on tasks requiring magnitude judgments, but retain the ability to recognize verbal and written numbers as well as Arabic numerals, suggests that the damaged (but not the preserved) systems reside in the inferior parietal cortex.

Brain imaging studies with normal adults reveal that tasks involving exact calculation recruit different brain areas from those involving approximate calculation. Specifically, the left and right parietal lobes show more activation for approximate than for exact calculation, while the left inferior frontal lobe shows the opposite pattern. And studies with bilingual subjects indicate that precise number facts are represented in the language in which they are learned, while approximate number facts, which engage subjects' sense of magnitude, seem to be represented in a format that is independent of language.

Taken together, this evidence provides strong support for the existence of distinct systems of numerical knowledge and processes. Note that two of the components in the model, the verbal number system and the Arabic numeral system, consist of knowledge that must be learned. The magnitude number system, however, is thought to be an innate, evolved capacity. If so, we would expect to find evidence of it in non-human animals and prelinguistic infants.

## NUMERICAL ABILITIES OF ANIMALS AND HUMAN INFANTS

Many studies have documented extensive numerical abilities across a wide range of warm-blooded vertebrate species. Many animals, including rats, pigeons, parrots, raccoons and chimpanzees, are able to respond on the basis of number of stimuli, whether the stimuli are visual or auditory events, objects in the world, or actions of the animal itself (such as presses of a lever), and whether the stimuli are simultaneously or sequentially presented. Moreover, animals trained to respond to number of one kind of stimuli show generalization to stimuli of other kinds. A model of a magnitude number mechanism was developed by Warren Meck and Russell Church, to account for animals' numerical abilities. On this model, the magnitude number mechanism can be thought of as a small container into which units of water can be added – one unit for each item counted. The subsequent fullness of the container represents the total number of items counted. The discriminability of two numbers relies on the proportionate, rather than the absolute, difference between their values. Consequently, it is easier to discriminate smaller numbers (e.g. 2 and 3) than larger numbers with the same absolute difference (e.g. 12 and 13).

Predictions derived from this model have been tested with prelinguistic humans to investigate the nature of their numerical abilities. The majority of infant studies conducted recently have used looking time as a dependent measure. There are two basic paradigms: habituation, and violation of expectation. In habituation studies, infants are presented repeatedly with a particular stimulus until they become familiar with it. When an infant's looking time decreases, according to a predetermined criterion, the infant is considered to be 'habituated', and is then presented with new instances of the habituated stimulus interspersed with instances of a novel stimulus; the infant's looking time to each is measured. If infants can discriminate between the habituated and the novel stimuli,

then their looking time should recover (i.e., increase) for the novel stimulus, but not for the habituated stimulus.

In violation-of-expectation studies, there is no habituation phase: the infant is simply presented with trials consisting of series of events, the outcome of which is sometimes an 'expected' one (e.g.  $1 + 1 = 2$ ) and sometimes an 'unexpected' one (e.g.  $1 + 1 = 1$ ). If infants have expectations about the outcome that should obtain in a particular event, then they should look longer at outcomes that do not match their expectation.

If human infants possess the same magnitude number system as do non-human animals, they should show similar numerical abilities. In particular, infants' ability to discriminate two numbers should depend on their proportionate, not their absolute, difference; and infants' numerical abilities should apply to a wide range of entities (auditory, visual, etc.).

Experiments have shown that infants are sensitive to numerical quantity. They can discriminate between both small and large numbers of items, and, as predicted by the model, their ability to discriminate two numbers depends on their proportionate difference – infants can discriminate 2 items from 4 items, 8 from 16, and 16 from 32, but under similar conditions fail to discriminate 8 items from 12 items or 16 from 24. Moreover, their numerical abilities are abstract: infants can enumerate visual objects, collections of objects, events (e.g. jumps of a puppet), and auditory stimuli.

Recent studies recording infants' event-related potentials – which measure activity across different areas of the brain in response to stimuli – reveal that when making numerical discriminations, infants' parietal cortex is highly activated. This is the area of the brain that is responsible for the magnitude sense of number in adults. Further studies have shown that infants can do more than discriminate numbers: their numerical representations can also be used to perform simple numerical operations such as addition and subtraction. When 5-month-old infants are shown an object placed on a stage and then hidden behind a screen, and then shown another object placed behind the screen, they expect two objects to be revealed when the screen is removed, and will look for longer if one or three objects are revealed. Similarly, if shown two objects that are then hidden behind a screen, and then shown one of them being removed, infants expect one object to remain and will look for longer if two are revealed behind the screen.

Similar studies conducted with non-human primates show that they possess similar abilities. Both

rhesus macaques and cotton-top tamarins respond as do human infants when shown addition and subtraction situations as described above: they look for longer at incorrect outcomes than at correct ones. For example, when rhesus monkeys were shown one eggplant placed out of sight in a box, and then shown another eggplant also placed in the box, they expected two eggplants to be in the box, and looked for longer if only one was revealed.

## ABILITY TO REASON ABOUT CONTINUOUS QUANTITY

Humans – and other animals – can represent continuous as well as discrete quantity: the height of a table, the volume of a piece of cake, the weight of a child, the duration of a sound. Research indicates that the ability to represent continuous quantities is also present from an early age. Infants under 6 months of age can discriminate between objects that differ in their amount of surface area. They can also represent the height of an object, the distance traversed by an object, and the duration of an event. Indeed, some experiments intended as investigations of infants' numerical discriminations (e.g. between different numbers of items) may have inadvertently been testing their ability to measure and distinguish continuous quantities (e.g. total area): in these studies, continuous quantities were confounded with discrete quantities (because a greater number of items always had a greater total surface area). However, recent studies testing infants' numerical abilities while strictly controlling for continuous variables have shown clearly that these abilities are distinct. Infants can represent both continuous quantities, such as the total surface area of elements in a display, and discrete quantities, such as the number of elements in a display.

Thus, from a very early age – long before any formal schooling, and even before any language comprehension – humans have distinct systems for quantitative reasoning. Empirical results with infants suggest at least two basic representational systems for quantity (discrete and continuous), but there may well be many more. It is implausible that all of our representations of continuous quantities are subserved by the same mental structures. For example, we would intuitively expect that our processes for representing an object's speed, the height of an object, the duration of an event, and the weight of an object are quite distinct. However, little is known about the cognitive systems that represent and reason about these kinds of continuous quantities. To obtain a richer understanding of

the full range of humans' quantitative abilities, and the relationships between these different quantitative systems, will require rigorous empirical investigation of these abilities.

## CONCLUSION

In conclusion, there is a great deal of evidence supporting at least three components of number processing. Neuropsychological evidence shows that verbal number knowledge, visual recognition of the Arabic numerals, and our sense of numerical magnitude, while interconnected in normal humans, are functionally and anatomically distinct. Moreover, the magnitude number mechanism is special in that it is an evolved mechanism that humans share with other animals, and that is independent of language and learning. It gives us our sense of quantity, allows us to understand relationships between different quantities of items, and underlies our ability to perform simple numerical operations. Empirical findings with adults,

prelinguistic infants, and nonhuman animals converge to support this claim. Further research is necessary to determine in more detail the functional parameters of the analog magnitude mechanism, and how all three systems interact throughout development to form human adults' numerical abilities.

## Further Reading

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